# LQG Regulator and Applications to Neural Control of Movement

F. Crevecoeur

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# LQG?

- **Linear**: Linear dynamics in state and control variables,
- **Quadratic**: Quadratic cost function in state and control variables,
- **Gaussian**: Assume that the noise variables have normal distributions ($X \sim N(\mu, \sigma^2)$).

# LQG in Equations

**Control System:**

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \xi_k, \\ y_k &= Hx_k + \omega_k, \end{aligned}$$

$x_k \in \mathbb{R}^n$, $u_k \in \mathbb{R}^m$ and $y \in \mathbb{R}^p$. The initial state is given ($x_1$).

**Cost Function:**

$$\begin{aligned} J_k(x_k, u_k) &= x_k^T Q_k x_k + u_k^T R u_k, & Q_k \geq 0, \quad k = 1, 2, ...N, \\ J_N(x_N) &= x_N^T Q_N x_N. & R > 0. \end{aligned}$$

**Noise Parameters:**

$$\xi_k \sim N(0, \Omega_\xi), \quad \omega_k \sim N(0, \Omega_\omega).$$

# Part I: Control

# Control Problem:

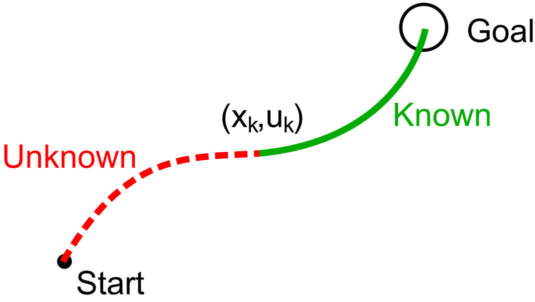
Find a control sequence, $u_1, u_2, \ldots u_{N-1}$, which minimizes:

$$J = E\left[J_N + \sum_{k=1}^{N-1} J_k(x_k, u_k)\right],$$

where $E(.)$ denotes the expected value of the arguments.

# How do we solve it? (1) Principle

If we know the optimal control sequence, $u_k, u_{k+1}, ...u_N$ with $1 < k < N$, then we should only solve the problem for $u_i$ with $1 \leq i < k$:



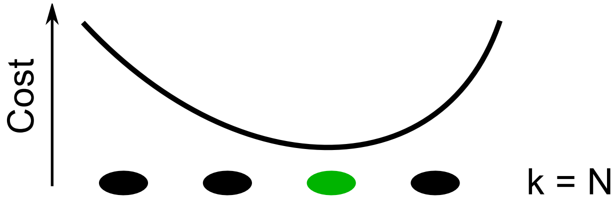
# How do we solve it? (2) Equations

We assume for now that the controller knows the exact state of the system $x_k$.
After some simple (which does not mean short) manipulations, one can show that the *cost-to-go* (i.e. the expected cost of the remaining trajectory) satisfies:

$$\begin{aligned} v_k(x_k, u_k) &:= \min_{u_k, u_{k+1}, ..} E\left[ J_N + \sum_{l=k}^{N-1} J_l(x_l, u_l) \right], \\ v_k(x_k, u_k) &= \min_{u_k} \left[ J_k(x_k, u_k) + E\left( v_{k+1} | x_k, u_k \right) \right]. \end{aligned}$$

This is the **Bellman** equation with initial condition: $v_N = x_N^T Q_N x_N$.

# Illustration of the Backward Recursion



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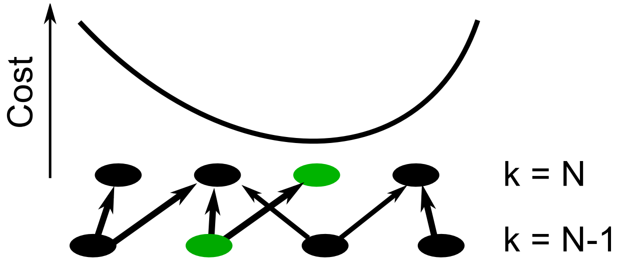

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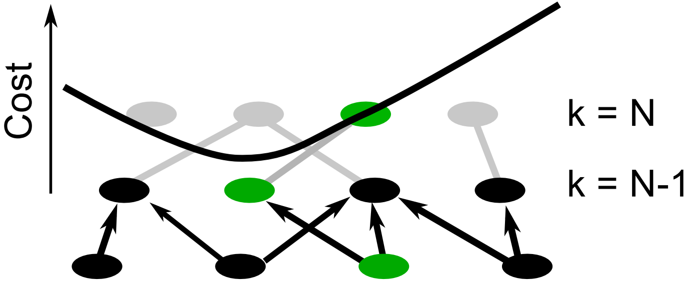
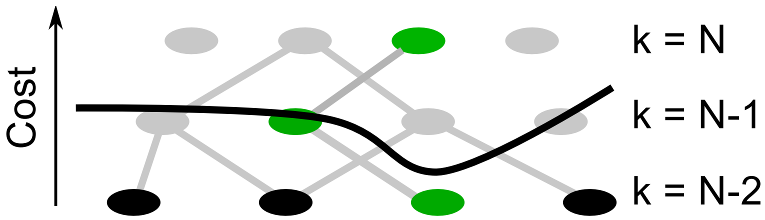
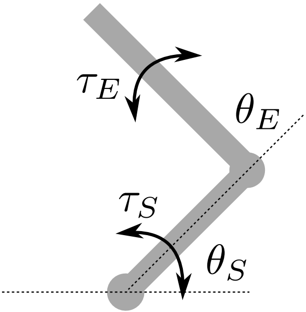


# Illustration of the Backward Recursion



# Curse of Dimensionality

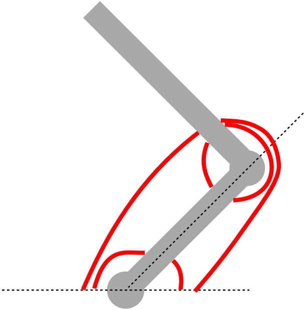



**Dynamic programming is practically useless:**

- State: $\text{Angles}_{S,E}$, $\text{Velocity}_{S,E}$, $\text{Torque}_{S,E}$, $\text{Torque Derivative}_{S,E}$
- Control $\text{Variables}_{S,E}$.

Considering 100 discretization points (coarse) and 50 time steps: $\sim$ h.

# Curse of Dimensionality (2)



**Dynamic programming is practically useless:**

- State: $\text{Angles}_{S,E}$, $\text{Velocity}_{S,E}$, $\text{Torque}_{S,E}$, $\text{Torque Derivative}_{S,E}$
- 6 Control Variables.

Considering 100 discretization points (coarse) and 50 time steps: $\sim$ yrs.

# Solution of the Optimal Control Problem

**Theorem: Fully Observable Case.** Under the optimal control policy, the cost-to-go is given by

$$v_k(x_k, u_k) = x_k^T S_k x_k + s_k,$$

with $S_k$ are positive semidefinite and and $s_k$ are non-negative.

# Solution of the Optimal Control Problem (2)

**Proof (Induction).**

- The claim is true when $k = N$ with $S_N = Q_N$ and $s_N = 0$.
- Let $1 \leq k < N$. We must solve:

$$v_k = \min_{u_k} \left[ x_k^T Q_k x_k + u_k^T R u_k + E\left(v_{k+1} | x_k, u_k\right) \right].$$

Expanding the conditional expected value of $v_{k+1}$ given $x_k$ and $u_k$ from the induction hypothesis gives:

$$\begin{aligned} v_k = \min_{u_k} \Big[ & x_k^T \left( Q_k + A^T S_{k+1} A \right) x_k + u_k^T \left( R + B^T S_{k+1} B \right) u_k \\ & + 2 x_k^T A^T S_{k+1} B u_k + \operatorname{tr}\left( S_{k+1} \Omega_\xi \right) + s_{k+1} \Big]. \end{aligned}$$

**Proof (Cont'd).**

The previous equation is a quadratic form in $u_k$, which is minimized when $u_k$ satisfies:

$$\begin{aligned} u_k &= -\left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A x_k, \\ &:= -L_k x_k. \end{aligned}$$

By plugging the expression of the optimal control variable into the expression of $v_k$ we obtain:

$$\begin{aligned} v_k &= x_k^T \left(Q_k + A^T S_{k+1}(A - BL_k)\right) x_k + s_k, \\ &:= x_k^T S_k x_k + s_k. \end{aligned}$$

where $s_k := s_{k+1} + \mathrm{tr}(S_{k+1}\Omega_\xi) > 0$. We found the required expression for $v_k$, and we must verify that $S_k \geq 0$ to complete the proof. ■

# Control: Practical Formulation

The optimal control policy is a linear function of the state. The optimal feedback gains are given by the following backward recursion:

$$
\begin{aligned}
L_k &= \left(R + B^T S_{k+1} B\right)^{-1} B^T S_{k+1} A, \\
S_k &= Q_k + A^T S_{k+1}(A - BL_k), \\
s_k &= s_{k+1} + \mathrm{tr}(S_{k+1}\Omega_\xi), \\
S_N &= Q_N, \quad s_N = 0.
\end{aligned}
$$

The closed loop control system is described by:

$$
x_{k+1} = (A - BL_k)x_k + \xi_k.
$$

# Comments

- The total expected cost under the optimal control policy is $v_1 = x_1^T S_1 x_1 + s_1$.
- The linear mapping of state into motor commands was not assumed a priori, it follows from linear dynamics and quadratic costs.
- Consider a one time step problem. $S_2 = Q_2$ and $u_1 = M Q_2 A x_1$ with $M$ adequately defined. From assumptions, $Q_2 \geq 0$. Assume there exists $1 \leq j \leq n$ such that $\lambda_j(Q_2) = 0$ (dimension of the null space $\mathrm{Ker}(Q_2)$ is $\geq 1$). If $Ax_1 \in \mathrm{Ker}(Q_2)$, then $u_1 = 0$. In other words, **if the system dynamics pushes the state in the null space of the constraints, the optimal control strategy is the unforced system**.

# Signal Dependent Noise

The variability of neural signal increases with the intensity of the signal. This is usually modelled by means of multiplicative noise:

$$x_{k+1} = Ax_k + Bu_k + \xi_k + \sum_{i=1}^{n_c} \varepsilon_i C_i u_k,$$

with $C_i$ scaling factors and $\varepsilon_i \sim N(0,1)$. In this case, the quadratic expression in $u_k$ contains an additional term that changes the optimal feedback gains as follows:

$$L_k = \left( R + B^T S_{k+1} B + \sum_{i=1}^{n_c} C_i^T S_{k+1} C_i \right)^{-1} B^T S_{k+1} A.$$

The signal dependent noise is at the origin of the **speed-accuracy trade-off**.

# Summary of Part I

- Solution of LQG control problems (problem definition, dynamic programming, backward recursion, induction proof).

**Application to neuroscience:**

- Uncontrolled Manifold, Minimum Intervention Principle $\Leftrightarrow$ Null Space of the State Constraint Matrices.
- Speed-Accuracy Trade-Off $\Leftrightarrow$ Signal Dependent Noise.

# Part II: Estimation

# Bayes Theorem

**Theorem (Bayes).** Let $A$ and $B$ be two events, the conditional probability of $B$ given $A$ is:

$$P(A|B) = \frac{P(A)P(B|A)}{P(B)}.$$

This can be applied to time varying stochastic processes. Let $x_t$ be a process and $y_t$ be a measurement of $x_t$. Assuming independent noise, the posterior distribution of $x_t$ is given by:

$$f(x_t|y_t) = \frac{f(x_t|y_{t-1}, y_{t-2}, \ldots)f(y_t|x_t)}{f(y_t|y_{t-1}, y_{t-2} \ldots)},$$

where the prior distribution is given by:

$$f(x_t|y_{t-1}, y_{t-2}, \ldots) = \int_{x_{t-1}} f(x_t|x_{t-1})f(x_{t-1}|y_{t-1}, \ldots)dx_{t-1}$$

# Kalman Filtering

**Theorem (Kalman Filtering).** Assume that (i) $x_0 \sim N(\mu_0, \Sigma_0)$, (ii) $x_t$ and $y_t$ satisfy:

$$
\begin{aligned}
x_{k+1} &= Ax_k + \xi_k \qquad \xi_k \sim N(0, \Omega_\xi) \\
y_k &= Hx_k + \omega_k \qquad \omega_k \sim N(0, \Omega_\omega),
\end{aligned}
$$

and (iii) $\xi_k$ and $\omega_k$ are independent, then we have $x_{k+1} \sim N(\mu_{t+1}, \Sigma_{t+1})$ where

$$
\begin{aligned}
\Sigma_{k+1|k} &= A\Sigma_k A^T + \Omega_\xi, \\
K_{k+1} &= \Sigma_{k+1|k} H^T \left( H\Sigma_{k+1|k} H^T + \Omega_\omega \right)^{-1} \\
\mu_{k+1} &= A\mu_k + K_{k+1}(y_{k+1} - HA\mu_k) \\
\Sigma_{k+1} &= (I - K_{k+1} H)\, \Sigma_{k+1|k}.
\end{aligned}
$$

# Alternative Approach: Predictive Case

Control System:

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \xi_k, \\ y_k &= Hx_k + \omega_k. \end{aligned}$$

We assume a convex combination of prior and feedback:

$$\begin{aligned} \hat{x}_{k+1} &= (1 - K) \times \text{prior} + K \times \text{feedback}, \\ \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K(y_k - H\hat{x}_k). \end{aligned}$$

The estimation error has the following dynamics:

$$e_{k+1} = (A - K_k H)e_k + \xi_k - K_k\omega_k.$$

**Predictive Case (Cont'd).**

The optimal Kalman gain minimize the estimation error:

$$\begin{aligned} K_k &= \arg\min_K \| e_{k+1} \|^2, \\ &= \arg\min_K \left[ \mathrm{tr} \left( E(e_{k+1} e_{k+1}^T) \right) \right]. \end{aligned}$$

From the error dynamics, the terms of the error covariance matrix that depend on $K_k$ give:

$$a(K_k) := \mathrm{tr} \left( -2K_k H \Sigma_k + K_k (H \Sigma_k H^T + \Omega_\omega) K_k^T \right),$$

which is minimized over $K_k$ when:

$$\nabla a(K_k) = 0 \quad \Rightarrow K_k = A \Sigma_k H^T (H \Sigma_k H^T + \Omega_\omega)^{-1}.$$

# Estimation: Practical Solution

The optimal state estimates and Kalman gains are obtained in a forward recursion ($\Sigma_1$ known):

$$
\begin{aligned}
\hat{x}_{k+1} &= A\hat{x}_k + Bu_k + K(y_k - H\hat{x}_k), \\
K_k &= A\Sigma_k H^T (H\Sigma_k H^T + \Omega_\omega)^{-1}, \\
\Sigma_{k+1} &= \Omega_\xi + (A - K_k H)\Sigma_k A^T.
\end{aligned}
$$

# Comments

- The control and estimation problems were solved independently and the induction proof is still valid with the state estimate (**verify it !**). This property is known as the **separation principle**. The closed loop dynamics corresponding to the full LQG solution is:

$$\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix} = \begin{bmatrix} A - BL_k & BL_k \\ 0 & A - K_k H \end{bmatrix} \begin{bmatrix} x_k \\ e_k \end{bmatrix} + \begin{bmatrix} \xi_k \\ \xi_k - K_k \omega_k \end{bmatrix}.$$

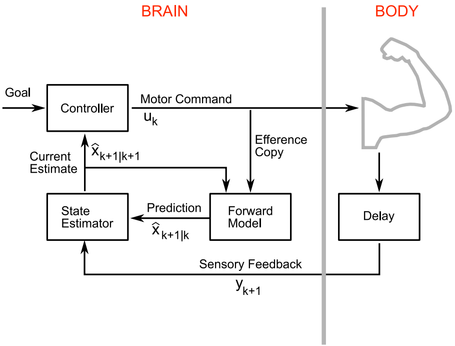
- The separation principle no longer holds in the presence of signal-dependent noise: need for simultaneous optimization of feedback and Kalman gains.

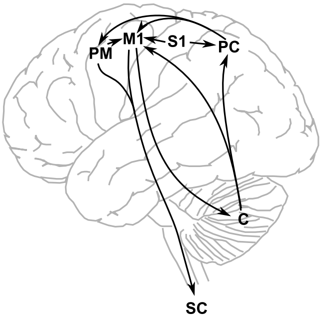
# Summary of Part II

- Kalman Filtering is Bayesian integration through time (maximum likelihood estimates, minimizes the estimation error).
- Control and estimation problems are solved independently. The stability of the closed loop dynamics depend of each problem independently.

# Part III: Applications

# Model & Neuroscience



# Translation of a Point Mass

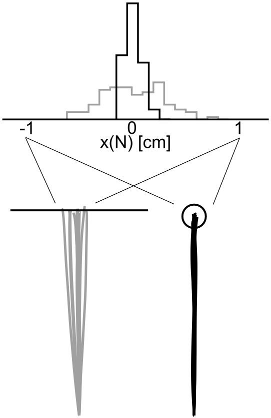
**Control System:**

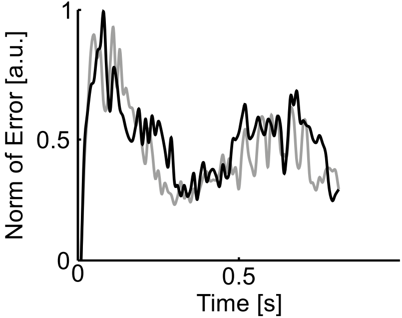
$$
\begin{aligned}
\ddot{x} &= -k_v \dot{x} + F_x, \\
\ddot{y} &= -k_v \dot{y} + F_y, \\
\tau \dot{F}_x &= u_x + \lambda u_y - F_x, \\
\tau \dot{F}_y &= u_y + \lambda u_x - F_y.
\end{aligned}
$$

- Additive noise, signal-dependent noise, prediction noise and delayed feedback (100 ms).
- Minimum intervention principle.
- Speed-accuracy tradeoff.
- Success and Variability.

# Minimum Intervention Principle

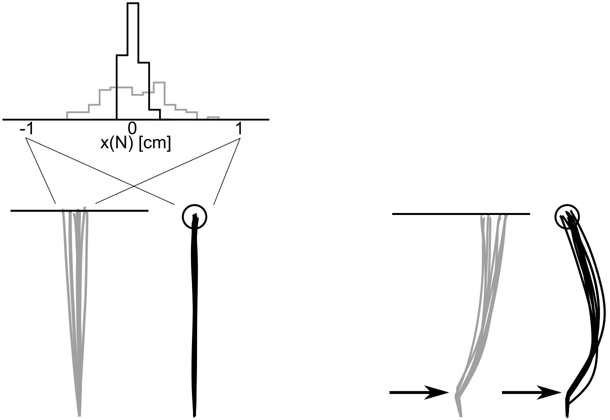
Translation of a point mass (10 cm) towards a dot (*x* and *y* constrained), or a bar (*y* constrained) within 700 ms:



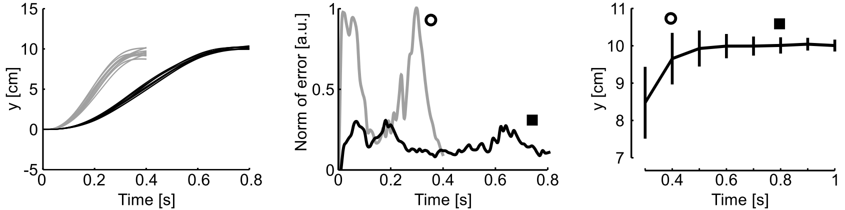
# Minimum Intervention Principle (2)

The perturbation along the unconstrained dimension is left uncorrected:



# Speed Accuracy Trade-Off: Fitt's Law

Fast movements are less accurate than slow movements:



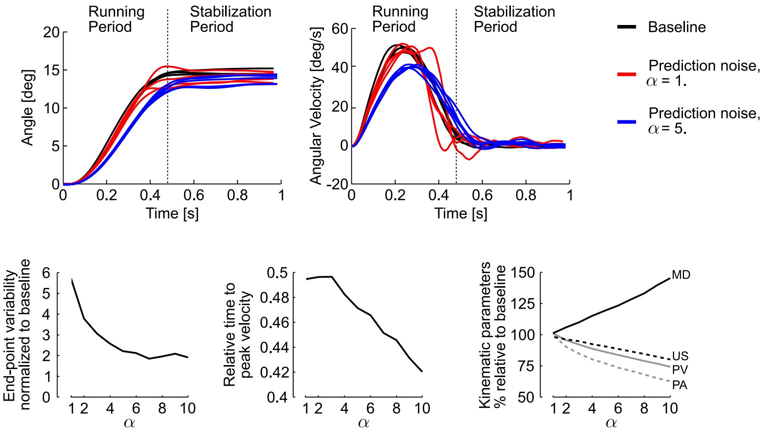
# Success and Variability

Prediction noise forces the system to rely more heavily on sensory feedback:

$$\begin{aligned} \hat{x}^p_{k+1} &= A\hat{x}_k + Bu_k + \eta_k, \qquad \eta_k \sim N(0, \Omega_\eta), \\ \hat{x}_{k+1} &= \hat{x}^p_{k+1} + K_{k+1}(y_{k+1} - H\hat{x}^p_{k+1}). \end{aligned}$$

# Success and Variability (2)

Adjusting the cost function ($Q_k/\alpha$ and $\alpha R$) restore smooth performance in spite of uncertainty in the internal model:



# Part VI: Math Reminders

# Tip # 1

A matrix $A \in \mathbb{R}^{n \times n}$ is positive definite if:

$$x^T A x > 0, \quad \forall x \in \mathbb{R}^n.$$

A matrix $A \in \mathbb{R}^{n \times n}$ is positive semidefinite if:

$$x^T A x \geq 0, \quad \forall x \in \mathbb{R}^n.$$

If $A > 0$, then $\lambda_i(A) > 0$ for $i = 1, 2, ...n$ and $A$ is invertible. For a positive semidefinite matrix $A$, there is a manifold $M$ embedded in $\mathbb{R}^n$ such that $Ay = 0$ for all $y \in M$. $M$ is called the null-space, or Kernel, of $A$.

# Tip # 2

- A probability space is a triple $(\Omega, \mathcal{U}, P)$, $\Omega$ being a set, $\mathcal{U}$ is a collection of subsets of $\Omega$ (called a $\sigma$-algebra) and $P$ is a measure of the elements of $\mathcal{U}$.
- $A \in \mathcal{U}$ is an event, and $P(A)$ is the probability of the event $A$.

$$P(A) := \int_A dP.$$

- The expected value of a random variable $X \in \Omega$ is:

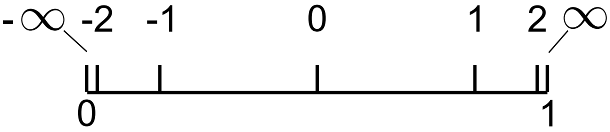
$$E(X) := \int_\Omega X dP.$$

- For a univariate Gaussian random variable, we have $\Omega = \mathbb{R}$, $\mathcal{U}$ is the collection of open and closed sets of $\mathbb{R}$ and the measure the Gaussian cumulative distribution function. The expected value is:

$$E(X) := \int_{\mathbb{R}} X f(X; \mu, \sigma^2) dX.$$

## Cont'd.

- A probability density function is a measure of a space. The Normal distributions maps de real numbers into [0 1] as follows:




- The uniform distribution between 0 and 1 is given by the Lebesgue measure.

# Tip # 3

The induction proof is a common tool to prove that $P(n)$ is true for all values of $n \in \mathbb{N}$.

- The initial case: show that $P(1)$ is true (easy, from assumptions).
- The induction case: assume that $P(n)$ is true with $n > 0$, show that it is also true for $P(n+1)$ (can be hard, very hard).
- If the induction case holds, than the set of $\bar{N} \subset \mathbb{N}$ such that $P(\bar{N})$ is false is between 1 and $n$. As $n$ is arbitrary, we have $\bar{N} = \emptyset$.

**Example:** All cars are the same colour.
**Proof:** One car is one colour. Any $n+1$ cars is made of two overlapping subsets of $n$ cars, with cars 1 to $n$ and 2 to $n+1$. From the induction hypothesis that any $n$ cars are the same colour, we have shown the the $n+1$ cars are the same colour. ■

# Tip # 4

**Lemma.** Let $x$ be a Gaussian random variable with mean value $\mu \in \mathbb{R}^n$ and covariance matrix $\Omega_x \in \mathbb{R}^{n \times n}$, and $S \in \mathbb{R}^{n \times n}$. Then

$$E(x^T S x) := \mu^T S \mu + \mathrm{tr}(S \Omega_x),$$

where tr(.) denotes the trace of the argument (i.e. the sum of the diagonal elements).

# Tip # 5

Let $f(x)$ be a real valued function whose derivatives up to order $n+1$ exist in the neighbourhood of $x_0$. The $n^{th}$ order Taylor's expansion of the $f(x)$ around $x_0$ is:

$$\begin{aligned} f(x) &\simeq \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \\ &\simeq f(x_0) + \left[\frac{df}{dx}\right]_{x_0}(x-x_0) + \frac{1}{2}\left[\frac{d^2 f}{dx^2}\right]_{x_0}(x-x_0)^2 + \ldots \end{aligned}$$

Euler integration is the application of Taylor's expansion to integration through time:

$$\begin{aligned} x(t+\delta t) &= x(t) + \dot{x}(t)\delta t + \mathcal{O}(\delta t^2), \\ &\simeq x(t) + f(x)\delta t. \end{aligned}$$

With linear dynamics, $f(x) = Ax$, we have:

$$x(t+\delta t) \simeq (I_n + A\delta t)x(t).$$

# Tip # 6

Any linear $n^{\text{th}}$ order ODE can be transformed in a $n$-dimensional first order ODE:

$$u^{(n)} = a_0 u + a_1 u' + \cdots + a_{n-1} u^{n-1}$$

$$\Leftrightarrow$$

$$\begin{bmatrix} u' \\ u'' \\ \vdots \\ u^{(n)} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ a_0 & a_1 & a_2 & \dots & a_{n-1} \end{bmatrix} \begin{bmatrix} u \\ u' \\ \vdots \\ u^{(n-1)} \end{bmatrix}$$

# Tip # 7

Expressing spatial constraints as a quadratic function is done by augmenting the system with the target vector. Let $x$ and $x^*$ be the state variable and the target. Then we have

$$\| x - x^* \|^2 = \begin{bmatrix} x & y & x^* & y^* \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ x^* \\ y^* \end{bmatrix}.$$

# Model Matrices

$$A_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -k_v & 0 & 1 & 0 \\ 0 & 0 & 0 & -k_v & 0 & 1 \\ 0 & 0 & 0 & 0 & \frac{-1}{\tau} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{\tau} \end{bmatrix} \quad B_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1/\tau & \lambda/\tau \\ \lambda/\tau & 1/\tau \end{bmatrix}$$

These matrices must be multiplied by $\delta t$ and augmented to include the target vector:

$$A = \begin{bmatrix} \mathcal{I}_{6\times 6} + \delta t A_0 & \mathcal{O}_{6\times 6} \\ \mathcal{O}_{6\times 6} & \mathcal{I}_{6\times 6} \end{bmatrix} \quad B = \delta t \begin{bmatrix} B_0 \\ \mathcal{O}_{6\times 2} \end{bmatrix}$$